Lecture 21: Newton's Method

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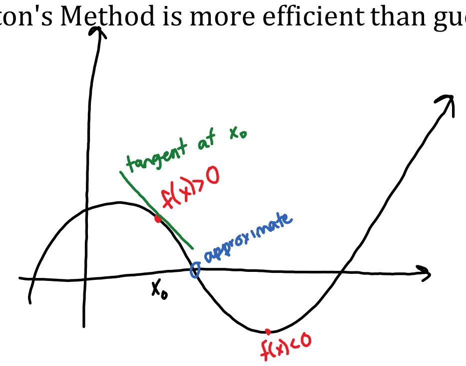
Newton's Method is an algorithm to find the zeroes of a function (approximately).

Recall: a zero of a function is a point $x = a$ where $f(a) = 0$ (where the graph of $f(x)$ crosses the x-axis.

Newton's Method is more efficient than guessing.



1) pick a starting point, x_0, close to the zero of f
2) compute tangent line at x_0
3) find the zero of tangent line
4) take that point as x_1, and use it as the new starting point

<u>Example</u>

$f(x) = x^3 - 2x - 5$ check for 2 values: $f(2) = 8 - 4 - 5 = -1$

$f(3) = 27 - 6 - 5 = 16$

$\rightarrow$ zero between 2 and 3

1 start at $x_0 = 2$: find tangent line

$f'(x) = 3x^2 - 2$

$$\begin{aligned} t_2(x) &= f'(2)(x-2) + f(2) \\ &= (3 \cdot 2^2 - 2)(x-2) + (-1) \\ &= 10x - 20 - 1 \\ &= 10x - 21 \end{aligned}$$

formula for tangent of f at a:

$t_a(x) = f'(a)(x - a) + f(a)$

2 find zero:

$$0 = 10x - 21$$

$$x = \frac{21}{10} = 2.1$$

back to 1 $x_1 = 2.1$: find tangent line

$$f'(2.1) = 3 \cdot \left(\frac{21}{10}\right)^2 - 2 = 11.23$$

$$f(2.1) = \cdots$$

$$\begin{aligned} t_{2.1}(x) &= f'(2.1)(x - 2.1) + f(2.1) \\ &= 11.23\,(x - 2.1) + f(2.1) \end{aligned}$$

$\ldots find\ zero\ (solve\ for\ x) \ldots$

$$0 = f'(2.1)(x - 2.1) + f(2.1)$$

$$-f(2.1) = f'(2.1)(x - 2.1)$$

$$-\frac{f(2.1)}{f'(2.1)} = x - 2.1$$

$$x = 2.1 - \frac{f(2.1)}{f'(2.1)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

recursive

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Newton's method

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$ Newton's method

<u>Example</u> find an approximation of $\sqrt[4]{3}$ (irrational number)

1) set $f(x) = x^4 - 3$, then $\sqrt[4]{3}$ is a zero of $f(x)$

$$0 = x^4 - 3$$
$$3 = x^4$$
$$x = \sqrt[4]{3}$$

2) Use Newton's method formula. We need a starting point and $f'(x)$

given: $x_0 = \frac{3}{2}$, $f'(x) = 4x^3$

$$x_1 = x_0 - \frac{f(x_0)}{f(x_0)} = \frac{3}{2} - \frac{\frac{33}{16}}{\frac{27}{2}} = 1.34722 = \frac{97}{72}$$

$$f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^4 - 3 = \frac{81}{16} - 3 = \frac{81 - 3.16}{16} = \frac{33}{16}$$

$$f'\left(\frac{3}{2}\right) = 4 \cdot \left(\frac{3}{2}\right)^3 = \cancel{4} \cdot \frac{27}{\cancel{8}\,2} = \frac{27}{2}$$

$$x_2 = \frac{97}{72} - \frac{f\left(\frac{97}{72}\right)}{f'\left(\frac{97}{72}\right)} = \cdots = 1.316074 = \sqrt[4]{3}$$

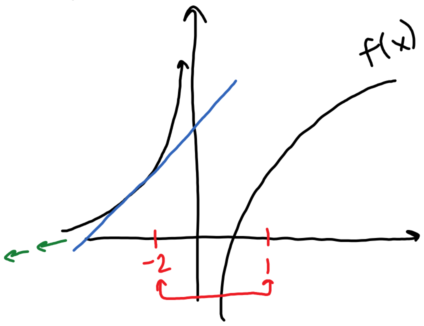
This method does not always work well. For example:

$f(x) = -\frac{1}{x} + 4$

$x_0 = 1$

$f'(x) = \frac{1}{x^2}$

$x_1 = 1 - \frac{f(1)}{f'(1)} = -2$



doesn't work!
zero of tangent line on the other side of discontinuity